A FRAGMENTATION PROBLEM FOR AN ELASTIC PLATE

N. I. Shishkin

We examine fragmentation in a linearly elastic plate subjected to the action of a moving load. The load is assumed to "run" at a "supersonic" constant velocity along one of the plate's sides. A loading of this type is encountered, for example, in explosive forming and explosive welding, during an oblique collision of two plates, etc. Fragmentation is understood as destruction of solid or liquid bodies occurring as a result of compression-wave reflection from free surfaces or from the interface between two bodies, one of which possesses a lower acoustic stiffness. Many interesting cases of fragmentation are described in Rinehart and Pearson's book [1], which also contains extensive literature references on the subject. Most of these papers give an acoustic interpretation of the fragmentation phenomenon with conditions of one-dimensional deformation. Lenskii [2] has tried to apply the theory of elasticity to the description of the fragmentation phenomenon in a plate experiencing a concentrated force. Ogurtsov [3] has obtained an exact solution to a stress problem for an elastic plate under a concentrated load; however, due to its complexity, his solution has yet to be applied to the fragmentation problem. In the present formulation of the problem, a simple exact expression for the stress waves in a plate, from which the fragmentation parameters can be determined, is obtained with relative ease. The solution can be used to obtain certain fragmentation data for a plate under a concentrated load, and also for an explosion in an elastic half-space. For example, it is possible to determine the approximate form of fragmentation cracks and formations near the epicenter.

1. Assume an elastic plate of thickness h referred to Cartesian coordinates x, y, and z. The lower free boundary of the plate lies in the plane $y=0$, and the upper in the plane $y=h$. The plate material is characterized by a density ρ and the propagation rates a and b of the elastic waves. Along the upper boundary of the plate there is, at a constant velocity $D>a$ a pressure wave whose front is parallel to the yz-plane

$$p = p_0 f\left[1/t_0\left(t - \frac{x}{D}\right)\right]. \tag{1.1}$$

Here, p_0 is the pressure in the wave front, t_0 the characteristic time of pressure variation in the wave, and f the shape of the wave which is assumed to be a diminishing function of its argument.

The equations of elastic theory, which describe the behavior of the plate in the elastic-displacement potentials φ and Ψ, have the form [4]

$$\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi, \qquad \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} = \Delta\psi, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.2}$$

Chelyabinsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 4, pp. 55-60, July-August, 1968. Original article submitted March 20, 1968.

At the surface of the plate y=h, the stress-tensor component σ_{yy} must be equal to the effective pressure, and the component σ_{xy} to zero:

$$\sigma_{yy} = \rho\left(1 - 2\frac{b^2}{a^2}\right)\frac{\partial^2\varphi}{\partial t^2} + 2\rho b^2\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\,\partial y}\right) = -p_0 f\left[\frac{1}{t_0}\left(t - \frac{x}{D}\right)\right] \qquad (y = h);$$

$$\sigma_{xy} = \rho b^2\left(2\frac{\partial^2\varphi}{\partial x\,\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) = 0 \qquad (y = h). \tag{1.3}$$

At the free boundary of the plate, we have

$$\sigma_{yy} = \sigma_{xy} = 0 \qquad (y=0). \tag{1.4}$$

Under the action of the load (1.1), a longitudinal P-wave and a transverse S-wave, whose fronts are shown in Fig. 1, propagate into the plate from the edge A of the load. Due to reflection from the free boundary y=0 of the plate, these waves generate new longitudinal and transverse PP-, SP-, SS- and PS-waves which now propagate toward the other plate boundary, where again they are reflected and generate new longitudinal and transverse waves which propagate toward the free boundary of the plate y=0. The process of reflection is repeated many times. In the area of the plate below the bottom of the load, there develops a network of elastic waves.

All waves whose fronts are shown in Fig. 1 are head waves and, therefore, the wave pattern depicted is to be observed only at those points of the plate which are sufficiently distant from the initial location of the load.

If load (1.1) is sufficiently intense, then even the reflection of the first longitudinal P-wave from the free boundary leads to a splitting off (fragmentation) of a part from the bulk of the plate. The possibility of fragmentation in a reflected transverse PS-wave and even in SP- and SS-waves is not excluded. In the following, we examine fragmentation under the assumption that it occurs in the first longitudinal PP-wave reflected from the free surface.

In conformity with this formulation of fragmentation, (1.2)-(1.4) may be divided into two simpler parts. In the first part, it is necessary to determine the potentials φ_0 and Ψ_0 of the waves generated by the moving load (1.1) in an elastic half-space $y \le h$, and in the second, to determine the potentials φ_1 and Ψ_1 of the waves reflected from the boundary of the half-space $y \ge 0$.

By comparing stresses which arise near the free boundary as a result of a superposition of the waves φ_1, Ψ_0, φ_1, Ψ_1 with the permissible stress, we can then locate the origin of the fragmentation cracks and determine the parameters of the fragmentations.

The potentials φ_0 and Ψ_0 must satisfy Eq. (1.2) and the boundary conditions (1.3); they have the form of plane waves

$$\frac{\varphi_0}{a^2 t_0^2} = -\frac{p_0}{2\rho b^2} W_p f_2(\tau_p), \qquad \frac{\psi_0}{a^2 t_0^2} = -\frac{p_0}{2\rho b^2} W_s f_2(\tau_s). \tag{1.5}$$

Here,

$$\tau_p = \frac{1}{t_0}\left(t - \frac{x}{D} - \frac{h-y}{a}\sqrt{1-m^2}\right), \qquad W_p = \left(\frac{1}{2}n^2 - m^2\right)\frac{1}{\Delta},$$

$$\tau_s = \frac{1}{t_0}\left(t - \frac{x}{D} - \frac{h-y}{a}\sqrt{n^2-m^2}\right), \qquad W_s = m\sqrt{1-m^2}\,\frac{1}{\Delta},$$

$$\Delta = \left(\frac{1}{2}n^2 - m^2\right)^2 + m^2\sqrt{1-m^2}\sqrt{n^2-m^2}, \qquad m = a/D, \qquad n = a/b,$$

$$f_2(\xi) = \int_0^{\xi} d\xi_1 \int_0^{\xi_1} f_2(x)\,dx\,.$$

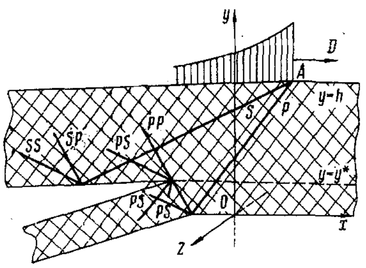

Fig. 1

The reflected-wave potentials φ_1 and Ψ_1 that satisfy Eq. (1.2) and the boundary conditions (1.4), written in terms of the potential sums $\varphi=\varphi_0+\varphi_1$ and $\Psi=\Psi_0+\Psi_1$, are as follows:

$$\frac{\varphi_1}{a^2t_0^2} = -\frac{p_0}{2\rho b^2}\left[W_pV_{pp}f_2(\tau_{pp}) - W_sV_{sp}f_2(\tau_{sp})\right]; \tag{1.6}$$

$$\frac{\psi_1}{a^2t_0^2} = -\frac{p_0}{2\rho b^2}\left[W_sV_{ss}f_2(\tau_{ss}) - W_pV_{ps}f_2(\tau_{ps})\right]. \tag{1.7}$$

Here,

$$V_{pp} = V_{ss} = \Delta_1/\Delta, \qquad V_{sp} = -2m\sqrt{n^2-m^2}\,({}^1/_2n^2-m^2)/\Delta,$$
$$V_{ps} = 2m\sqrt{1-m^2}\,({}^1/_2n^2-m^2)/\Delta,$$
$$\Delta_1 = -({}^1/_2n^2-m^2)^2 + m^2\sqrt{1-m^2}\sqrt{n^2-m^2},$$
$$\tau_{pp} = \frac{1}{t_0}\left(t-\frac{x}{D}-\frac{h+y}{a}\sqrt{1-m^2}\right),$$
$$\tau_{sp} = \frac{1}{t_0}\left(t-\frac{x}{D}-\frac{h}{a}\sqrt{n^2-m^2}-\frac{y}{a}\sqrt{1-m^2}\right),$$
$$\tau_{ss} = \frac{1}{t_0}\left(t-\frac{x}{D}-\frac{h-y}{a}\sqrt{n^2-m^2}\right),$$
$$\tau_{ps} = \frac{1}{t_0}\left(t-\frac{x}{D}-\frac{h}{a}\sqrt{1-m^2}-\frac{y}{a}\sqrt{n^2-m^2}\right).$$

In (1.6)-(1.7) the quantities V_{ij} (i, j = p, s) are the reflection coefficients of waves reflected from the free boundary of an elastic half-space. The subscripts p and s refer to a longitudinal and a transverse wave, respectively; the first subscript denotes an incident wave and the second, a reflected wave.

The stressed state near the free surface of the plate in the space between the fronts of the reflected PP- and PS- waves is characterized by the following (other than zero) stress-tensor components:

$$\frac{\sigma_{xx}}{2\rho b^2} = \left(\frac{1}{2}n^2-1\right)\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\,\partial y} = -\frac{p_0}{2\rho b^2}W_p\left(\frac{1}{2}n^2+m^2-1\right)[f^+]$$
$$\frac{\sigma_{yy}}{2\rho b^2} = \left(\frac{1}{2}n^2-1\right)\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\,\partial y} = -\frac{p_0}{2\rho b^2}W_p\left(\frac{1}{2}n^2-m^2\right)[f^+]$$
$$\frac{\sigma_{zz}}{2\rho b^2} = \left(\frac{1}{2}n^2-1\right)\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{p_0}{2\rho b^2}W_p\left(\frac{1}{2}n^2-1\right)[f^+]$$
$$\frac{\sigma_{xy}}{2\rho b^2} = \frac{\partial^2\varphi}{\partial x\,\partial y} + \frac{1}{2}\frac{\partial^2\psi}{\partial y^2} - \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} = \frac{p_0}{2\rho b^2}W_pm\sqrt{1-m^2}\,[f^-]$$
$$[f^+] \equiv f(\tau_p) + V_{pp}f(\tau_{pp}), \qquad [f^-] \equiv f(\tau_p) - V_{pp}f(\tau_{pp}). \tag{1.8}$$

2. Fragmentation parameters will be determined with the aid of Huber's destruction criterion (see, for example, [5, 6]): if $p \equiv 1/3(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) > 0$, then the destruction of the material occurs at the point where the specific-potential elastic-strain energy has its maximum value; if $p<0$, then destruction occurs at the point where the specific strain energy of distortion has its critical value. The specific potential-strain energy for the plane tensile strength $\frac{1}{2}\sigma_*^2/E$ is taken as the critical value of the energy. We get

$$\frac{1-2\nu}{6E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})^2 + \frac{1+\nu}{6E}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 +\right.$$
$$\left.+ (\sigma_{zz}-\sigma_{xx})^2 + 6\sigma_{xy}^2\right] = \frac{\sigma_*^2}{2E} \quad \text{for } p>0,$$
$$\frac{1+\nu}{6E}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6\sigma_{xy}^2\right] = \frac{\sigma_*^2}{2E}$$
$$\left(\nu = \frac{n^2-2}{2(n^2-1)}\right) \quad \text{for } p<0; \tag{2.1}$$

where σ_* is the tensile strength of the material corresponding to the critical value of the total specific strain energy; ν is Poisson's ratio; E is Young's modulus.

By substituting (1.8) into (2.1), we get the fragmentation condition as

$$A\,[f^+]^2 + B\,[f^-]^2 = \frac{4}{W_p^2}\frac{\sigma_*^2}{p_0^2} \qquad (p>0),$$

$$A_1\,[f^+]^2 + B_1\,[f^-]^2 = \frac{4}{W_p^2}\frac{3}{2(1+\nu)}\frac{\sigma_*^2}{p_0^2} \qquad (p<0),$$

$$A \equiv \frac{3n^2-4}{n^2-1}\,[n^2-1+(1-2m^2)^2] = \frac{3n^2-4}{n^2-1}(n^2-1+\cos^2 2\alpha)$$

$$B \equiv \frac{3n^2-4}{n^2-1}\,4m^2(1-m^2) = \frac{3n^2-4}{n^2-1}\sin 2\alpha$$

$$A_1 \equiv 1+3\cos^2 2\alpha, \qquad B_1 \equiv 12m^2(1-m^2) = 3\sin^2 2\alpha\,. \tag{2.2}$$

In a loading wave where the pressure in the wave front experiences a jump and diminishes behind the front, fragmentation can occur only at the front of a wave reflected from the free surface. Thus, the moment of fragmentation by a PP-wave coincides with moment of arrival of the front of the PP-wave at the point (in the medium) of interest to us, i.e.,

$$t^* = \frac{x}{D} + \frac{h+y^*}{a}\sqrt{1-m^2} \qquad (\tau_{pp}=0). \tag{2.3}$$

By eliminating t* from (2.2) on the basis of the foregoing considerations, we arrive at an equation for determining the dimensionless coordinate of a fragmentation crack $\eta^*=y^*/at_0$

$$f^2(2\eta^*\cos\alpha) - 2C_i f(2\eta^*\cos\alpha) + F_i = 0 \qquad (i=1,\,2); \tag{2.4}$$

where i=1 corresponds to p > 0; i=2 to p < 0

$$C_1 = \left[-1-\frac{1-2\nu}{2(1-\nu)}(1-\cos 4\alpha)\right]V_{pp}(\alpha,\nu);$$

$$F_1 = V_{pp}^2(\alpha,\nu) - \frac{1-2\nu}{1-\nu^2}\frac{\sigma_*^2}{p_0^2 W_p^2(\alpha,\nu)}; \tag{2.5}$$

$$C_2 = -\frac{1+3\cos 4\alpha}{4}V_{pp}(\alpha,\nu), \qquad F_2 = V_{pp}^2(\alpha,\nu) - \frac{\sigma_*^2}{2p_0^2 W_p^2(\alpha,\nu)}\frac{3}{1+\nu};$$

$$\eta^* = y^*/at_0, \qquad \cos\alpha = \sqrt{1-m^2}. \tag{2.6}$$

3. Where the loading wave has a "triangular" profile

$$f(t/t_0) = \begin{cases} 1-t/t_0 & (0\leqslant t\leqslant t_0) \\ 0 & (t\geqslant t_0), \end{cases} \tag{3.1}$$

(2.4) takes the form

$$(1+2\eta^*\cos\alpha)^2 - 2C_i(1-2\eta^*\cos\alpha) + F_i = 0 \qquad (i=1,\,2).$$

Hence, for a dimensionless fragmentation crack, we have

$$\eta_i^* = \frac{1}{2\cos\alpha}\left(1-C_i\pm\sqrt{C_i^2-F_i}\right) \qquad (i=1,\,2). \tag{3.2}$$

Equation (3.2) becomes appreciably simpler for D=∞ (simultaneous short impact against one of the plate's sides). In this case

$$\alpha=0, \quad p>0, \quad W_p=1, \quad V_{pp}=-1, \quad C_1=1$$

$$F_1 = 1-\frac{1-2\nu}{1-\nu^2}\frac{\sigma_*^2}{p_0^2}, \qquad \eta^* = \frac{y^*}{at_0} = \frac{1}{2}\left(\frac{1-2\nu}{1-\nu^2}\right)^{1/2}\left|\frac{\sigma_*}{p_0}\right|, \tag{3.3}$$

i.e., the fragmentation thickness is directly proportional to the length of the reflected wave and to the dynamic tensile strength of the material, is inversely proportional to the wave amplitude, and depends on Poisson's ratio.

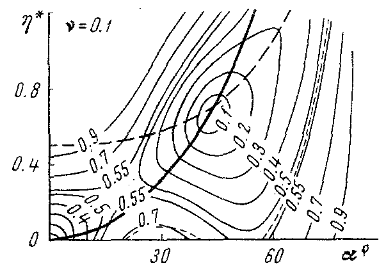

Fig. 2a

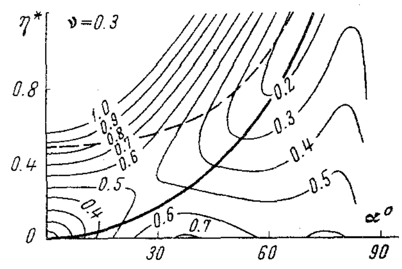

Fig. 2b

The maximum fragmentation thickness is that of the entire plate, and, therefore, $y^* \leq h$. Furthermore, from (3.1) and $0 \leq f \leq 1$, it follows that

$$0 \leqslant 1 - 2\eta^* \leqslant 1, \qquad \eta^* \leqslant 1/2, \qquad y^* \leqslant 1/2 \; at_0 .$$

This means that the fragmentation thickness in any material does not exceed a half wavelength. One arrives at a well known result: in order that fragmentation occur in the first reflected longitudinal wave, the wavelength must be shorter than double the plate thickness.

Figure 2 gives the results of an evaluation of the function $\eta^*(\alpha, \nu)$ from (3.2) for $\nu=0.1$ and $\nu=0.3$ as a function of the angle of incidence.

The $\eta^*(\alpha, \nu)$ lines shown in Figure 2 can be treated as isolines of the parameter $\sigma^\circ \equiv |\sigma_*/p_0|$ (on them, $\sigma^\circ(\eta, \alpha)=\text{const}$), or as isolines of the quantity $1/2\, \sigma_*^2/E$ –the potential strain energy that corresponds to the tensile strength. The dashed line in Figure 2 represents the curve

$$\eta^* = 1/(2 \cos \alpha) \tag{3.4}$$

above which fragmentation cannot occur in virtue of the inequality

$$0 \leqslant f\,(2\eta^* \cos \alpha) \leqslant 1.$$

The heavy solid line in Figure 2 corresponds to the curves

$$p = 1/3\; (\sigma_{xx} + \sigma_{yy} \times \sigma_{zz}) = 0. \tag{3.5}$$

At the points in the plane $\eta^*\alpha$ above (3.5), the plate material experiences tension at arrival of a PP-wave, while below these curves it experiences compression.

The finely dashed lines in Figure 2 (these lines come to lie in the region $p < 0$) are those where the strain energy of distortion in P+PP waves is equal to that at the front of an incident P-wave. In the region between the finely dashed lines and (3.5), destruction in accordance with (2.1) must be recognized as impossible since, according to the assumptions, there is no destruction in a P-wave. Possible fragmentation regions in PP-waves are, firstly, the region in which $p > 0$ and, secondly, those portions of the region $p < 0$ in which the strain energy of distortion is greater than that of the incident-wave front. The latter regions lie between the finely dashed lines and the axis $\eta^* = 0$ (Figure 2, a). For Poisson's ratio $\nu \geq 0.26$, such regions do not exist. For example, for $\nu=0.3$, they are absent (Figure 2 b). This means that at the arrival of a PP-wave front at plate points which lie in the region $p < 0$ of the plane $\eta^*\alpha$, the strain energy of distortion at these points is smaller than that at the incident-wave front. Hence, for such values of ν, fragmentation in the region $p < 0$ is impossible.

Let us examine the manner in which the fragmentation thickness in a PP-wave changes with a change in the incidence angle α of a longitudinal P-wave onto the free surface.

For "small" values of the parameter σ° ($\sigma^\circ \lesssim 0.5$), the fragmentation thickness decreases and tends to zero as the angle α varies from zero to a certain value that depends on σ°. The concavity of the curve $\eta^*(\alpha)$ points downward (see, for example, the curves $\sigma^\circ=0.4$ in Fig. 2). For such σ°, multiple fragmentation is possible, since above these σ° curves, the destruction criterion is all the more fulfilled.

For large values of $\sigma°$ ($\sigma° \gtrsim 0.5$), the fragmentation thickness increases with increasing α, and the concavity of the curve $\eta^*(\alpha)$ points upward. For sufficiently large α (for example, for $\alpha = 23°$ and $\sigma° = 0.6$ in Figure 2 a) the fragmentation thickness is equal to the plate thickness. In this case, the entire plate appears to be a "fragmentation." With a further increase in α, fragmentation becomes possible only where the point (η^*, α, $\sigma°$) comes to lie in the region where the strain energy of distortion is greater than that at the front of the incident P-wave. As previously noted, this is possible only for Poisson's ratio $\nu \lesssim 0.26$. For $\nu \gtrsim 0.26$, fragmentation below the line p=0, in conformity with (2.1), is not possible.

For certain values of the parameter $\sigma°$ and angle α (for example, for $\sigma° = 0.5$, $\alpha = 13°$, see Figure 2a) two values of the fragmentation thickness are possible. From these values, the lower one should be taken, since a reflected wave appears first at those points of the plate that are closer to the free surface.

Calculation of the isolines $\sigma°$=const were performed also for other values of Poisson's ratio. Their graphical pattern is analogous to the one given in Fig. 2.

The plot of the fragmentation thickness vs. the incidence angle of a P-wave with the free surface of the plate, as shown in Figure 2, was obtained under the assumption that fragmentation occurs in a PP-wave. This assumption is justified for small α, where the reflection coefficient V_{pp} (α, ν) is large compared to V_{ps} (α, ν). However, with increasing α, the absolute value of the V_{pp} (α, ν) coefficient decreases and even vanishes at a certain angle (polarization-exchange angle [7]) that depends on ν. It is obvious, that for α close to the polarization-exchange angles, fragmentation in a PP-wave cannot occur. From an analysis of this situation, it may be seen that fragmentation is possible in a reflected transverse PS-wave. Consideration of PS-waves is imperative also for free-surface points. This is the reason why the sections of the isolines in Figure 2 are not exact near the x-axis.

4. Plots of the computed fragmentation thickness vs. the wave incidence angle, shown in Figure 2, are suitable for predicting the possible profiles of fragmentation cracks near the zero point of a concentrated explosion in an elastic half-space. Approximate profiles of such cracks can be obtained by rotating the sections of the curves (Figure 2) near to the axis η^* about this axis.

The profiles of the cracks are highly dependent on the parameter $\sigma^0 = |\sigma_*/p_0|$.

For small values of $\sigma°$ (which corresponds to a low tensile-strength medium or to an explosion near the free surface), fragmentation cracks must have a bowl-shape whose concavity faces the free surface.

Vice versa, for large values of $\sigma°$ (which correspond to a high tensile-strength medium or an explosion occurring deep within a half-space), the fragmentation cracks must have a bowl-shape whose concavity faces the explosion.

For intermediate values of $\sigma°$ ($\sigma° \approx 0.5$), cracks parallel to the free surface are possible.

The shape of the crack profiles at the epicenter of an explosion (or impact) is an indication of possible fragmentation formations. Thus, in addition to fragmentations in the form of planoconvex lenses there can occur fragmentations in the form of concavoconvex and planoconcave lenses.

The shape and dimensions of fragmentation cracks and individual fragments depend on both on the parameter $\sigma°$ and Poisson's ratio, as well as on the wavelength αt_0.

In conclusion, the author wishes to express his gratitude to Silkin for his discussions and to Buslov for performing the computations.

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